Stochastic Subspace Projection Schemes for Solving Random Mistuning Problems in Jet Engine Bladed-Disks: A New, Fast and Accurate Stochastic Perturbation Matrix Reduced-Order Modeling (SPM ROM) Approach

Dr. Dan M. Ghiocel
Ghiocel Predictive Technologies, Inc.
6 South Main St., 2nd Floor,
Pittsford, NY 145345
Email: dan.ghiocel@ghiocel-tech.com
Http://www.ghiocel-tech.com

Abstract:

Stochastic subspace projection schemes provide efficient ways for building stochastic physics-based reduced-order models for accurately solving large-size stochastic mechanics problems. The paper introduces a new, fast and accurate approach for predicting large (and small) random mistuning in bladed-disks using an efficient reduced-order modeling (ROM). The proposed ROM approach is based on a stochastic subspace projection scheme called herein the Stochastic Perturbation Matrix (SPM) approach. The shown results, plus other in-house numerical tests, indicate that the SPM ROM approach can outperform other mistuning ROM approaches in terms of accuracy and computational performance. The author believes that the SPM ROM approach will play a gradually increasing role in mistuning predictions, and in a relatively short time period will become a popular choice for solving large mistuning problems.

1.0 Basic Concepts of Stochastic Finite Element Methods

The problem of simulating stochastic physical systems can be often reduced to that of finding the stochastic solution u, for a linearized stochastic PDE operator L, and a applied stochastic forcing function Q that has the general form

$$\begin{aligned} &L(u,t) + Q = 0 \text{ in } \Omega \\ &u = \bar{u} \text{ on } \partial\Omega_d \\ &B(u) = \bar{u} \text{ on } \partial\Omega_n \end{aligned} \quad (1)$$

where $\partial\Omega_d$ is the boundary of the domain for which the solution or primary variable is specified (i.e., Dirichlet boundary conditions) and $\partial\Omega_n$ denotes the boundary with natural (i.e., Neumann boundary conditions) for which the dual variable (e.g., force) is specified.

Complex stochastic physical phenomena are modeled, with complex stochastic PDEs, which may include domains of entirely different physical behavior (as with fluid-structure interaction problems), or may have multiple coupled physical fields (as with thermal-mechanical systems). Thus the actual simulation may be a composition of multiple field domains and these PDEs can become highly complex. Also, accurate simulations are rarely linear, though often they are linearized for use in a nonlinear analysis solution (e.g., using Newton's method). The stochastic finite element method is a way of formulating, or (spatially) discretizing, stochastic partial differential equations in simple algebraic system of equations that can be efficiently solved numerically. The stochastic finite element method provides a sound method of finding the "optimal" solution, within a given stochastic subspace in which the true stochastic solution belongs. Commonly the Galerkin scheme is used for optimality criteria, that is to find $\tilde{u} \in S = \text{span}(\phi_i)$, i = 1, …, n such that $L\tilde{u} + Q \perp S$ or equivalently $\langle (L\tilde{u} + Q), v\rangle = 0 \ \forall v \in S$ where $\langle a, b\rangle = \int_\Omega a \cdot b$.

2.0 Stochastic Perturbation Matrix Approach

Using finite element method, a linearized PDE can be transformed in a n x n linear algebraic system of equations:

$$\mathbf{K}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \tag{1}$$

in which **u** is the solution of the problem, **K** is the generalized stiffness matrix of the system and **F** is the generalized loading function. All the three terms of the equation (1) are functions of the stochastic input vector, **x**. If the system is stochastic, then the stiffness matrix **K** is a stochastic matrix. For a realization k of the stochastic input vector, $\mathbf{x}_k$, the stochastic stiffness matrix variation with respect to the mean stiffness matrix (that is assumed to correspond to the input mean vector, $\bar{\mathbf{x}}$) can be computed by:

$$\Delta\mathbf{K}(\mathbf{x}_k, \bar{\mathbf{x}}) = \mathbf{K}(\mathbf{x}_k) - \mathbf{K}(\bar{\mathbf{x}}) \tag{2}$$

where $\mathbf{K}(\mathbf{x}_k)$ and $\mathbf{K}(\bar{\mathbf{x}})$ are the stochastic stiffness matrix computed for realization k of the stochastic input vector, $\mathbf{x}_k$, and the mean stiffness matrix, respectively. For a random realization k of the stochastic input vector, equation (1) can be rewritten as follows:

$$[\mathbf{K}(\bar{\mathbf{x}}) + \Delta\mathbf{K}(\mathbf{x}_k, \bar{\mathbf{x}})]\mathbf{u}(\mathbf{x}_k) = \mathbf{F}(\bar{\mathbf{x}}) \tag{3}$$

For the sake of simplicity, and without any loss in generality, it is assumed that the generalized loading function is a deterministic quantity. Further, if the stochastic stiffness variation in realization k is proportional to the mean stiffness by a scalar factor $\varepsilon(\mathbf{x}_k, \bar{\mathbf{x}})$, then equation (2) can be rewritten:

$$\Delta\mathbf{K}(\mathbf{x}_k, \bar{\mathbf{x}}) = \varepsilon(\mathbf{x}_k, \bar{\mathbf{x}})\mathbf{K}(\bar{\mathbf{x}}) \tag{4}$$

Typically, for structural mechanics problems, the stochastic scale (or proportionality) factors $\varepsilon(\mathbf{x}_k, \bar{\mathbf{x}})$ are in the order of few percents, or even less. These proportionality factors describe non-dimensionally, in percents, the stochastic system variation, or in other words, the stochastic system perturbation with respect to the mean system.

More generally, for a non-proportional stochastic variation of the system, the generalized perturbation matrix can be defined by:

$$\boldsymbol{\varepsilon}(\mathbf{x}_k, \bar{\mathbf{x}}) = \mathbf{K}^{-1}(\bar{\mathbf{x}})\Delta\mathbf{K}(\mathbf{x}_k, \bar{\mathbf{x}}) \tag{5}$$

The Stochastic Perturbation Matrix (SPM) approach provides an efficient solution to stochastic equation (3) by projecting it onto the reduced-size subspace whose basis vectors are computed by the products of the generalized perturbation matrix, $\boldsymbol{\varepsilon}(\mathbf{x}_k, \bar{\mathbf{x}})$, risen at increasing powers (such as the terms of a power series) with the mean solution vector, $\mathbf{u}(\bar{\mathbf{x}})$.

Thus, the SPM basis vectors are defined by the matrix products

$$\boldsymbol{\varepsilon}(\mathbf{x}_k, \bar{\mathbf{x}})^{j-1}\mathbf{u}(\bar{\mathbf{x}}), \ j = 1 \text{ to } s \tag{6}$$

in which s is the number of the SPM basis vectors or the size of the SPM subspace.

The SPM solution of the stochastic equation (3) is:

$$\mathbf{u}(\mathbf{x}_k) = \sum_{j=1}^{s} y_j(\mathbf{x}_k, \overline{\mathbf{x}})\, \boldsymbol{\varepsilon}(\mathbf{x}_k, \overline{\mathbf{x}})^{j-1} \mathbf{u}(\overline{\mathbf{x}}) \qquad (7)$$

in which the coefficients y_j are unknown quantities. They need to be determined before using the above equation (7). These coefficients are computed extremely fast by solving a very reduced system of equations of size s x s, s << n, that is defined by the similarity transformation of original system from the original space to the SPM subspace. It should be noted that the SPM basis vectors belong to the well-known family of Krylov-Arnoldi-Lanczos vectors that are popular in numerical analysis for fast solving of large-size linear systems.

3.0 Application of the SPM ROM Approach to Compute the Complex Frequency Mistuned Response

The stochastic equation of motions of a bladed-disk system in complex frequency domain can be written:

$$[\overline{\mathbf{K}} + \Delta\mathbf{K} + i\omega(\mathbf{C} + \mathbf{G}) - \omega^2(\overline{\mathbf{M}} + \Delta\mathbf{M}) + \mathbf{Z}_a]\mathbf{u} = \mathbf{f}_e \qquad (8)$$

Using a compact notation for dynamic complex stiffness term, equation (9) can be rewritten:

$$\left[\widetilde{\mathbf{K}}(\overline{\mathbf{x}}) + \Delta\widetilde{\mathbf{K}}(\mathbf{x}, \overline{\mathbf{x}})\right]\mathbf{u}(\mathbf{x}) = \mathbf{f}_e(\overline{\mathbf{x}}) \qquad (9)$$

The stochastic solution of equation (9) employing the SPM ROM approach is similar to equation (7) but is formulated in the complex frequency domain:

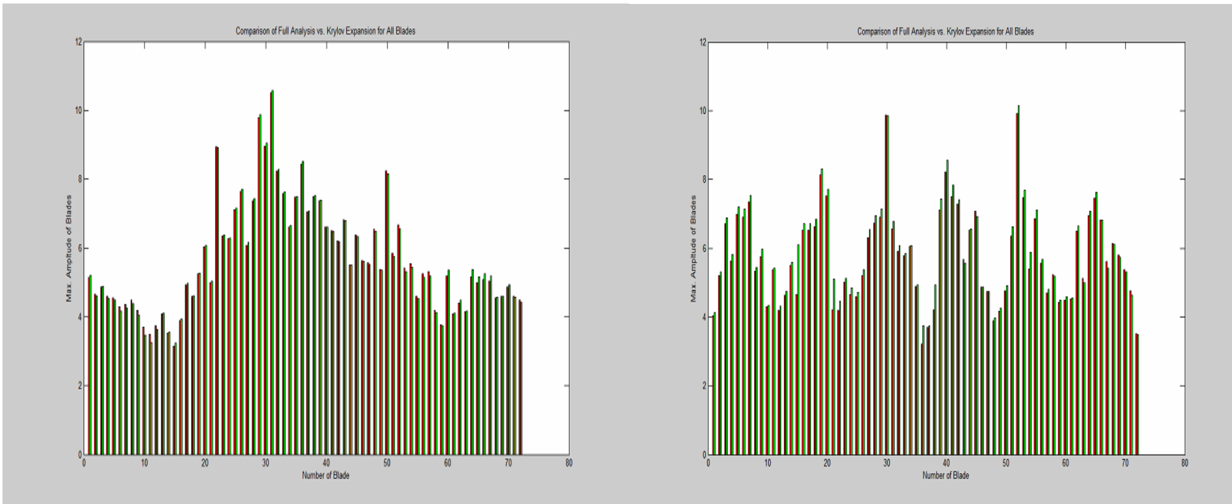
$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{s} y_j(\mathbf{x}, \overline{\mathbf{x}})\boldsymbol{\varepsilon}(\mathbf{x}, \overline{\mathbf{x}})^{j-1}\mathbf{u}(\overline{\mathbf{x}}) \qquad (10)$$

in which the coefficients y_j are computed fast by solving a reduced complex system of size s x s, s << n, for each frequency of interest. For bladed-disk mistuning problems, the reduced-order system size s varies typically from 5 to 25 equations. To be applied efficiently, the SPM ROM solution in equation (10) has to be applied for all frequencies of interest using the results of a *single* modal analysis solution for the tuned (mean geometry) system. The complex frequency response of the tuned system can be computed for all frequencies at any location based on the tuned mode participation factors and mode shapes.

The advantage of the proposed SPM ROM approach versus other random mistuning ROM approaches resides in its clean theoretical basis and assumptions with no restriction for large mistuning problems and no artificially added model constraints. The SPM ROM approach is perfectly fitted for solving large mistuning problems, for both the low-order and high-order system modes, including complex mode couplings in veering regions, where other mistuning ROM approaches may produce less accurate results.

4.0 Illustrative Example of A 72 Blade Compressor Blisk System

To illustrate the capability of the proposed SPM ROM approach, a 72 blade compressor blisk model was considered. Figure 1 shows the comparative results for the first blade bending mode of the mistuned system using a direct, full-model (exact) analysis and the SPM ROM approach. Figures 1a and 1b show the mistuned blade tip amplitudes for different engine orders of the aero-forcing function.



a) Engine Order 2 b) Engine Order 3

Figure 1. Blade tip amplitude computed using full-model analysis (red) and SPM ROM approach (green)

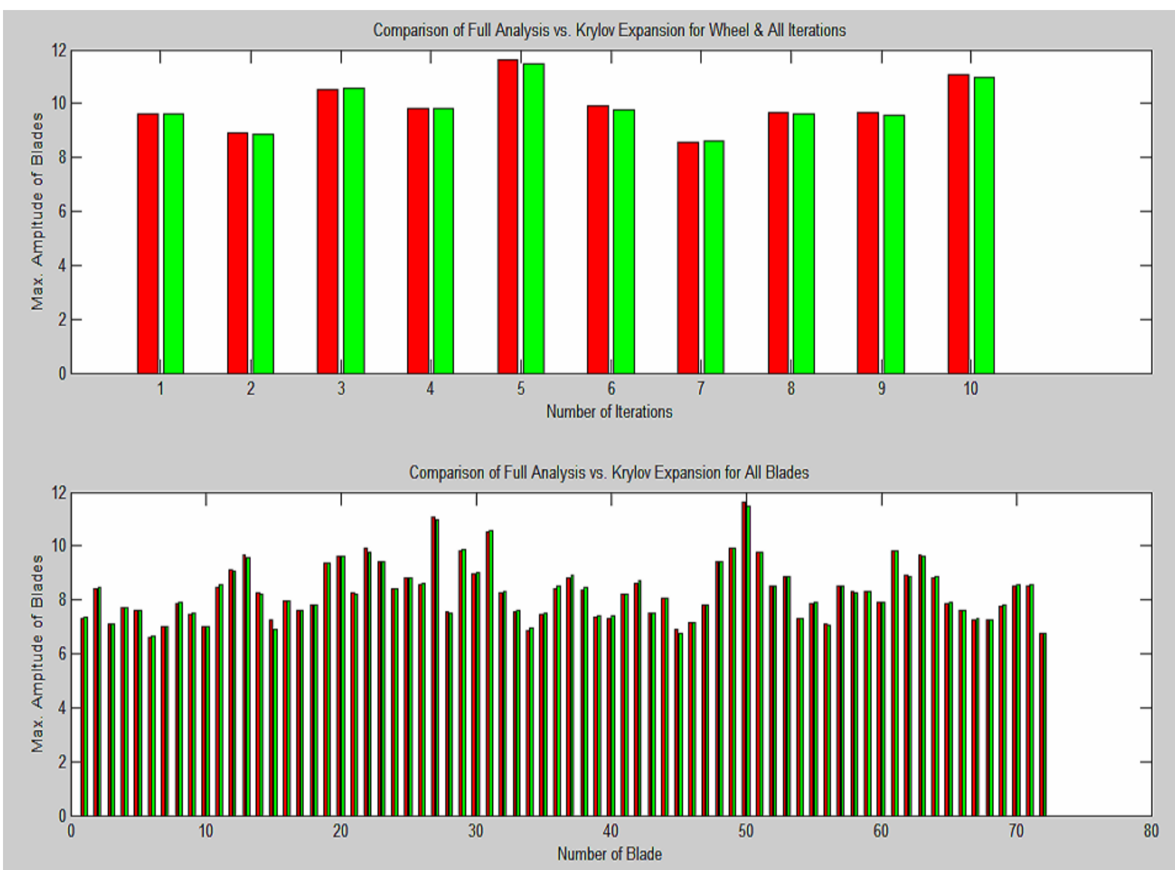
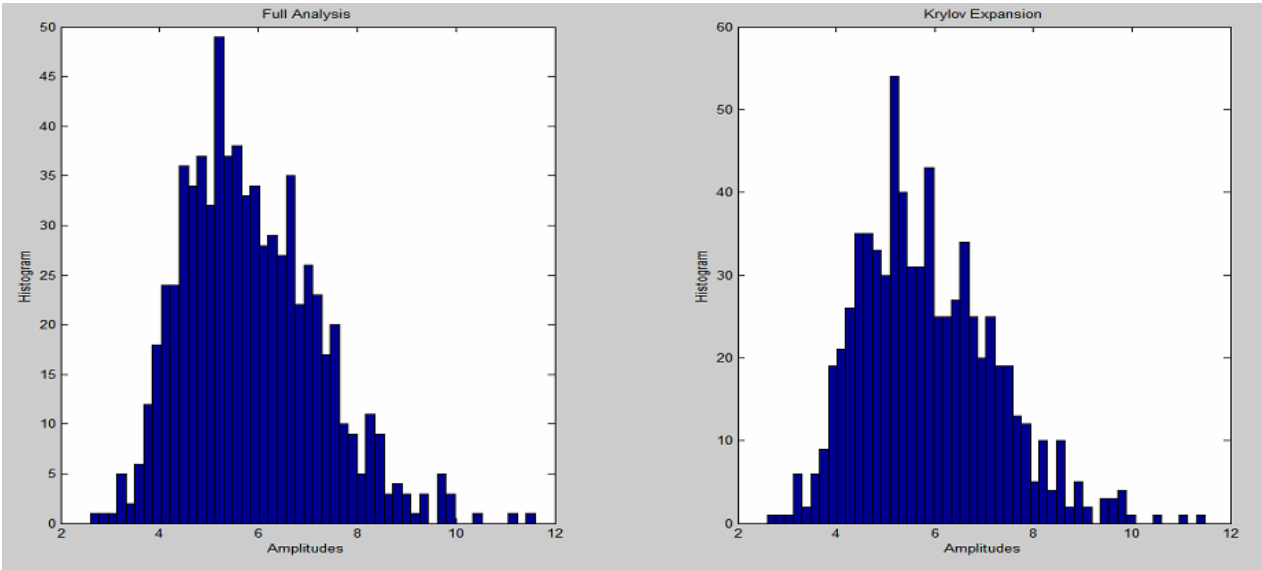


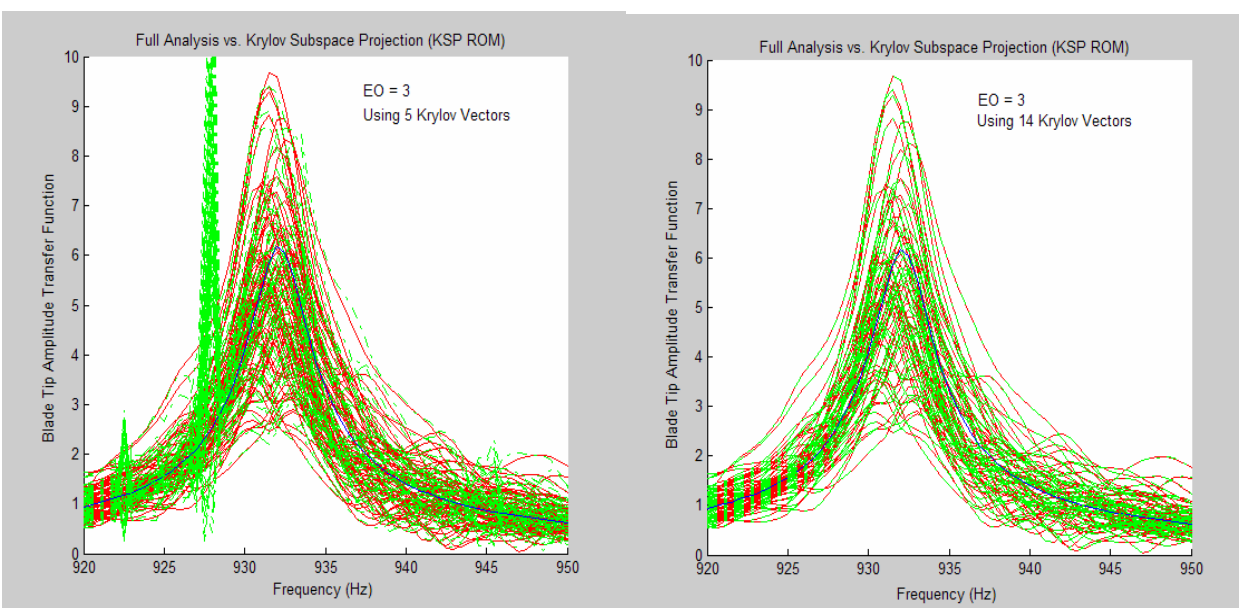
Figure 2. Blade tip responses, per simulation (upper plot) and per blade location (lower plot), computed using full-model analysis (red) and SPM ROM approach (green) for ten realizations of stochastic input

Figure 2 shows the maximum blade tip amplitude, per simulation and per wheel (maximum over all system blades) and per blade (maximum over all simulations), respectively, computed using full-model (exact) analysis and the SPM ROM approach.



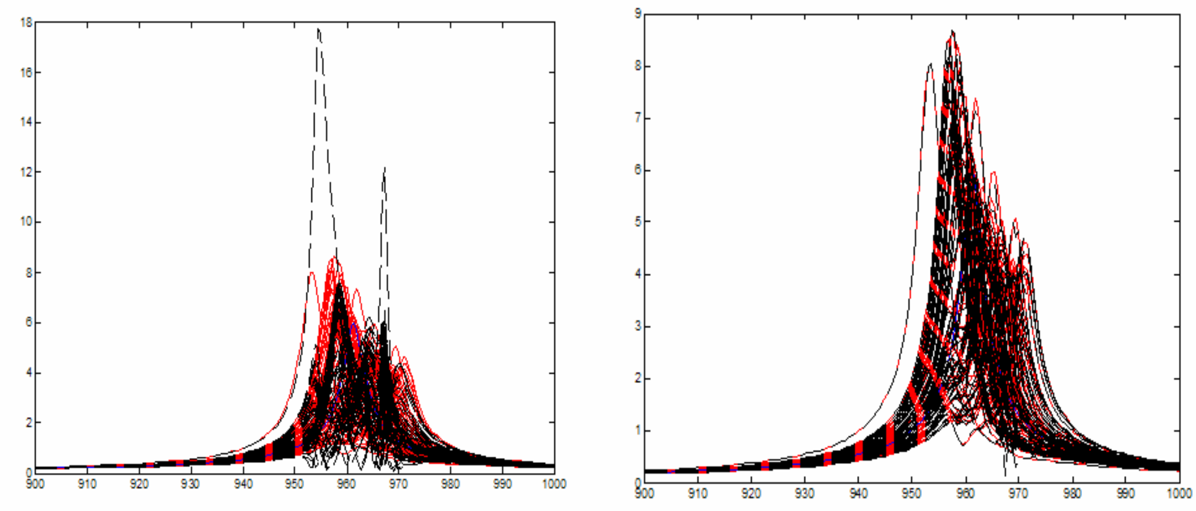
a) Using full-model analysis　　　　b) Using SPM ROM approach

Figure 3. Computed blade tip response histograms for for full-model analysis and SPM ROM



a) 5 equation ROM size　　　　b) 14 equation ROM size (converged)

Figure 4. Computed blade tip amplitudes using the full-analysis model and the SPM ROM approach, with a 5 equation ROM size or 5 basis vectors, and a 14 equation ROM size or 14 basis vectors, respectively. Original Blisk Design (red solid line is the full-analysis, green dashed line is the SPM ROM).



a) 5 equation ROM size

b) 14 equation ROM size (converged)

Figure 5. Computed blade tip amplitudes using the full-analysis model and the SPM ROM approach, with a 5 equation ROM size or 5 basis vectors, and a 14 equation ROM size or 14 basis vectors, respectively. Modified Blisk Design (red solid line is the full-analysis, black dashed line is the SPM ROM).

Figure 3 shows the computed histograms of blade tip amplitudes in the first blade bending mode for a set of 7,200 simulations using the full-model analysis (exact) and the SPM ROM approach. Figure 4 illustrates the blade tip amplitude transfer function for the first blade bending mode computed using SPM ROM for 5 basis vectors and for 14 basis vectors, i.e. ROM size of only 5 and 14 equations, respectively.

Figure 5 illustrates the same blade tip amplitude transfer functions computed using SPM ROM for 5 basis vectors and for 14 basis vectors, i.e. ROM size of 5 and 14 equations, respectively, but for a modified blisk design with a stiffer disk. It should be noted that the results showing in Figures 4 and 5 indicates the the 72 blade mistuned blisk response can be accurately captured by employing a SPM ROM of a size of only 14 equations.

Figures 6 through 8 show the use of the SPM ROM approach in combination with response surface modeling for reducing random mistuning responses in blisks. A practical application of this kind is the optimal repair of FOD-damaged IBRs in depots. The SPM ROM approach is capable of handling accurately the large geometry mistuning problems as those of FOD-damaged IBRs.

The mistuned response of the 72 blade blisk system in the first blade bending mode is shown in Figure 6. As shown in the figure, the maximum mistuned response occurs at the blades 22, 17 and 63. Our interest is on how to change the stiffness of blade 22, or blade 17, or blade 63 to reduce the mistuned response. Figure 7 shows the response surfaces of the blade tip amplitude of blade 22 assuming that this is a function of the stiffness variations of blades 22 and 63, and blades 22 and 17, respectively. All the other blades were considered to have the same mistuned stiffnesses (fixed perturbed geometries) as those that were used for the simulation shown in Figure 6.

Figure 8 shows the response surfaces of the maximum blade amplitude around the blisk wheel assuming that this is a function of the stiffness variations of blades 22 and 63, and blades 22 and 17, respectively. All the other blades were considered to have the same mistuned stiffnesses (fixed perturbed geometries) as those that were used for the simulation shown in Figure 6.

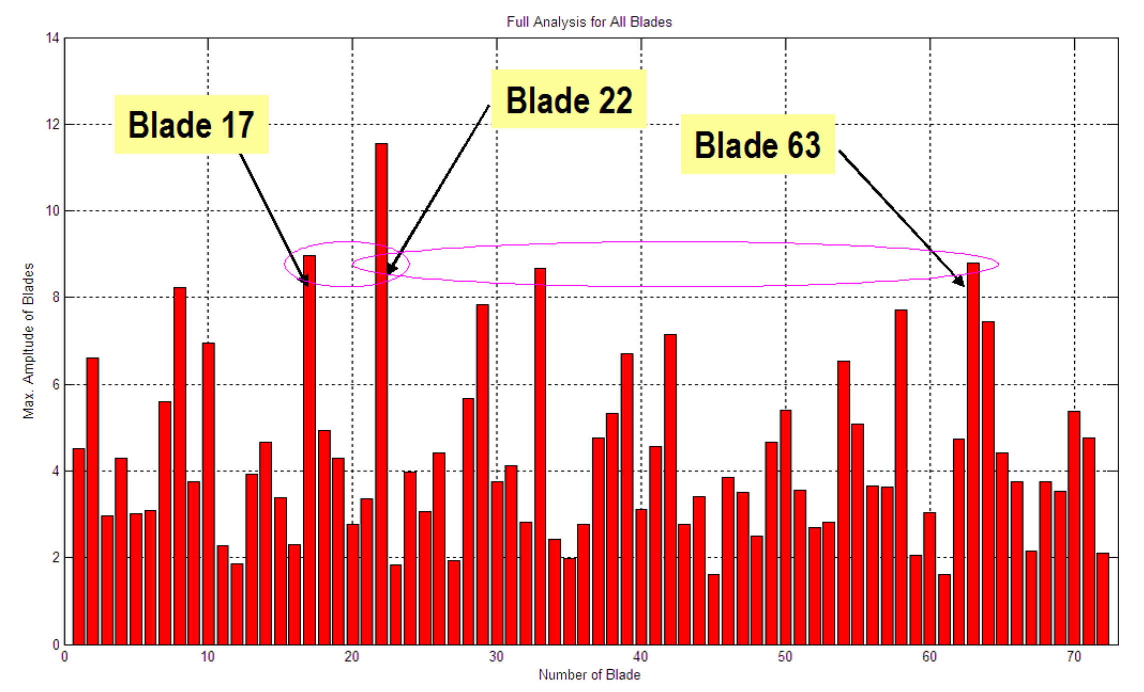


Figure 6. Simulated mistuned blade tip amplitude pattern for the 72 blade blisk system

Both Figures 7 and 8 illustrate that the mistuned response surfaces are highly nonlinear with respect to blade stiffness variations (solution points are also shown). A two-level hierarchical approximation was employed for modeling the mistuned response surfaces (Ghiocel, 2004, 2005).

It should be noted that the current industry practice of building response surface approximation using quadratic regression using DOE sampling rules fails shortly, as indicated in Figure 7. The low-amplitude response areas can be totally missed if a typical quadratic regression DOE-based approach is employed. The same conclusion is valid for the results shown in Figure 8 that illustrates the response surface of the maximum response over the blisk wheel. Figure 8 also shows that the effect of varying the stiffness of blade 22 is much more significant on the maximum mistuned response than the effect of varying the stiffness of blade 63, and it is about the same with the effect of varying the stiffness of blade 17.

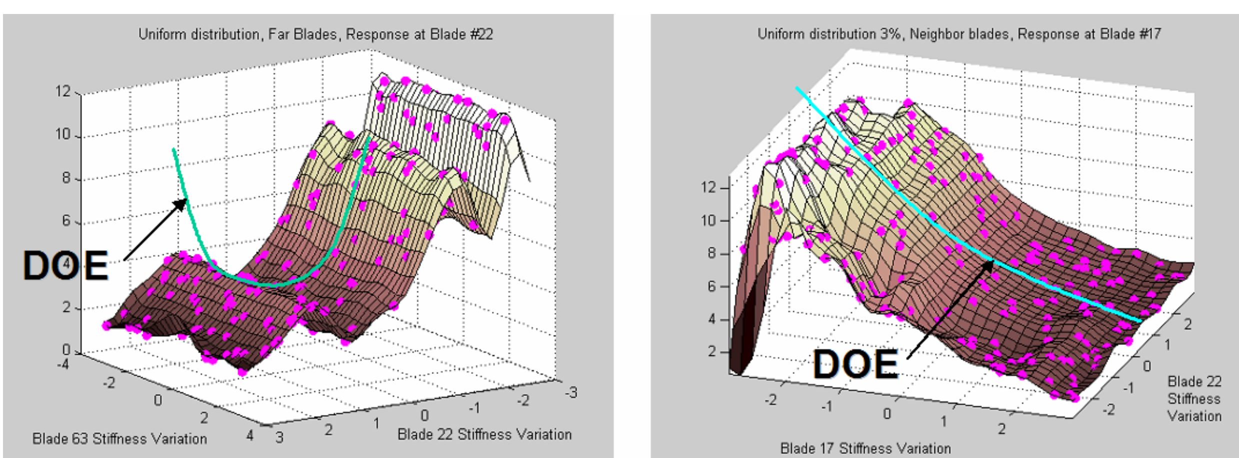


Figure 7. Response surfaces of the blade tip amplitudes for blades 22 as a function of stiffness variation of blade pair 22 and 63, and 22 and 17, respectively (see Figure 6).

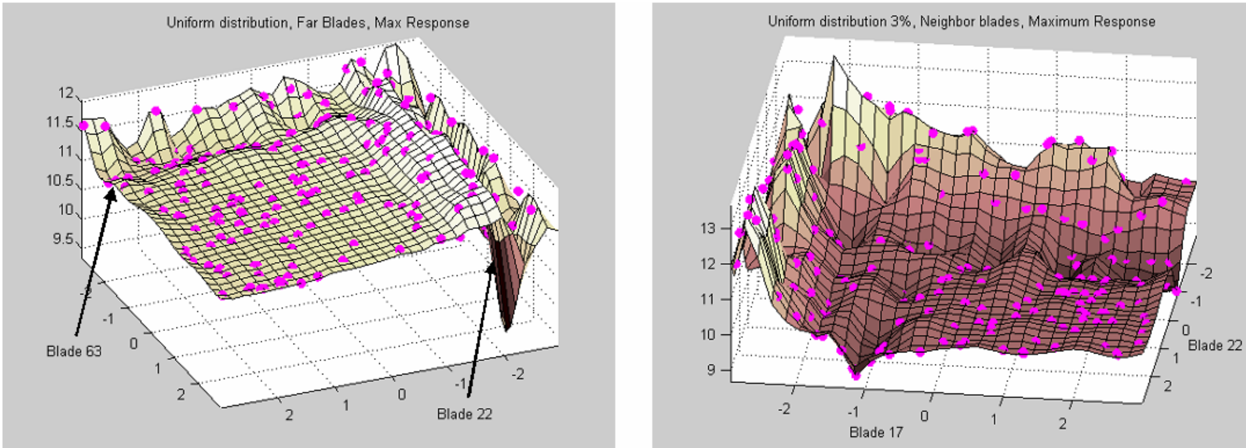


Figure 8. Response surfaces of the maximum blade tip amplitude around the blisk wheel as a function of the stiffness variation of the blades of the pairs 22 and 63, and 22 and 17, respectively (see Figure 6)

5.0 Concluding Remarks

The paper proposed a fast and accurate approach for solving large (and small) mistuning problems in engine bladed-disks. As shown herein, a potential application of the proposed approach can be the optimal repair of FOD-damaged IBRs. The results shown indicate that the SPM ROM approach can outperform other mistuning ROM approaches in terms of accuracy and computational performance (very reduced ROM size). For the illustrated 72 blade compressor disk case study, the required size of the SPM ROM system is of only 14 equations. To simulate mistuned responses only a *single* deterministic modal analysis using the full rotor model is required for computing the tuned system modes.

The SPM ROM approach is perfectly fitted for solving large mistuning problems, for both low-order and high-order system modes, including complex dynamic couplings in veering regions. Due to its capability of accurately handling large mistuning problems with a high computational performance, the author believes that the SPM ROM approach will play a gradually increasing role in future mistuning prediction and will become a popular choice for solving large mistuning problems.

The paper also shows that the current industry practice of building response surface approximations using quadratic regression based on DOE sampling rules fails shortly for mistuning problems.

6.0 Acknowledgements

This original work is a part of an internal research project of GP Technologies, Inc. The intention of this paper is to simulate new research ideas for solving large (and small) mistuning problems in bladed-disks for the benefit of the aircraft jet engine technology.

7.0 References

1. Ghiocel, D.M., "Stochastic Response Surface Approximation Using (Bayesian and Fuzzy) Hierarchical Models", Presentation at *the 2005 National HCF Conference,* New Orleans, LO, March 14-18, 2005

2. Ghiocel, D.M., "Stochastic Simulation Methods for Engineering Predictions", *Chapter 20 in the "Engineering Design Reliability Handbook"*, eds. Nikolaidis, Ghiocel and Singhal, CRC Press, December, 2004